Dropping a particle out of a roller coaster

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Abstract
A rider in a roller coaster lets go of a particle such as a small marble. How far does the marble travel horizontally from the point of release before hitting the ground, assuming the speed of the roller coaster is determined by conservation of mechanical energy starting from the highest hill up which it was pulled? Where should the marble be released along the track if one wishes to maximize the range of the marble? These questions constitute interesting variations on conventional problems in two-dimensional kinematics, appropriate for an undergraduate course in classical mechanics. Exploration of various shapes of tracks could form interesting student projects for numerical or experimental investigation.

Keywords: roller coaster, maximum range, two-dimensional kinematics, plane curves

1. Statement of the problem

A particle starts at rest at the origin and is transported frictionlessly in a roller coaster [1] along a curve $y(x)$ with the x-axis pointing horizontally in the direction of motion and the y-axis pointing vertically downward, as sketched in figure 1. The particle is dropped out of the cart when it is at point (x, y) and falls freely thereafter, starting from rest relative to the cart. At the release point, the speed of the particle relative to the ground is $\upsilon = \sqrt{2gy}$. Denote by θ the angle at which it is launched relative to the horizontal; $\theta > 0$ if it is launched downward toward the ground, and $\theta < 0$ if upward. If the ground is at height h below the origin, then the particle is at altitude $h - y$ above the ground when it is released. What is the horizontal distance travelled by the particle after it is released? What release point along the curve maximizes that range?

2. General solution

Horizontally the particle travels an additional distance X during its flight time t, where

$$X = (\upsilon \cos \theta)t. \tag{1}$$

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14. ABSTRACT
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15. SUBJECT TERMS

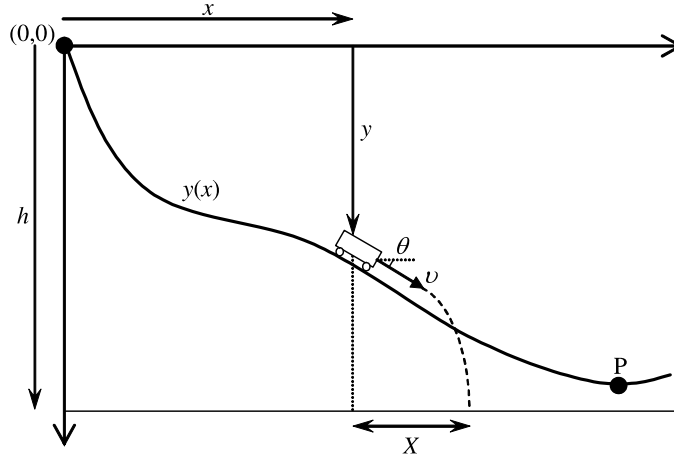| 16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
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Figure 1. A particle dropped from rest relative to a cart and following the dashed trajectory to the ground. The speed of the particle with respect to the track is υ, as a result of the kinetic energy gained from the gravitational potential energy lost. Both friction and air drag are neglected. Point P marks a local minimum of the curve.

Vertically the particle will reach the ground when

$$h - y = (\upsilon \sin\theta)t + \tfrac{1}{2}gt^2 \tag{2}$$

so that the flight time is

$$t = \frac{-\upsilon \sin\theta + \sqrt{\upsilon^2 \sin^2\theta + 2g(h-y)}}{g} \tag{3}$$

where the positive solution has been taken of the quadratic equation (2), because extrapolation to a time before $t = 0$ is not of interest. Substituting this result back into equation (1) along with $\upsilon = \sqrt{2gy}$, one obtains

$$X = -2y \sin\theta \cos\theta + 2\cos\theta \sqrt{hy - y^2 \cos^2\theta}. \tag{4}$$

Divide this equation through by h to make the expression dimensionless, by measuring x, y, and X in units of h,

$$X = -2y \, \mathrm{sgn}(\sin\theta) \cos\theta \sqrt{1 - \cos^2\theta} + 2\cos\theta \sqrt{y - y^2 \cos^2\theta}. \tag{5}$$

However

$$\cos\theta = \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}x}{\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}} = \frac{1}{\sqrt{1 + y'^2}} \tag{6}$$

(which can alternatively be derived from $\sec^2\theta = 1 + \tan^2\theta$) where $y' \equiv \mathrm{d}y/\mathrm{d}x$, so that

$$X = 2\frac{\sqrt{y(1 - y + y'^2)} - yy'}{1 + y'^2}. \tag{7}$$

As a check on this result, if the particle is launched at $y = 0$, then equation (7) implies that $X = 0$, which is correct because the projectile is then released at the origin from rest relative to the ground and it falls straight downward. If it is instead launched at $y = 1$ (i.e. at $y = h$ in dimensional units) with a downward slope, equation (7) again implies that $X = 0$, which is correct assuming the particle cannot burrow underground. In contrast, if it is launched at

$y = 1$ with an upward slope (due to an upturned lip at the end of the curve) such that $y' = -k$ for some positive k, then equation (7) becomes

$$X = \frac{4k}{1 + k^2}. \tag{8}$$

To see that this result is correct, use the standard range formula

$$X = \left| \frac{\upsilon^2 \sin 2\theta}{g} \right|. \tag{9}$$

However, $\upsilon^2 = 2g$ at the dimensionless value $y = 1$, so that

$$X = |4 \sin\theta \cos\theta| = 4\sqrt{\frac{\tan^2\theta}{1 + \tan^2\theta}} \frac{1}{\sqrt{1 + \tan^2\theta}} \tag{10}$$

which becomes equation (8) using $\tan\theta = -k$.

3. Examples for specific shapes of tracks

3.1. Particle launched horizontally

Suppose the curve has a local minimum or maximum at the release point, such as point P in figure 1, so that $y' = 0$ there. Then

$$X = 2\sqrt{y(1 - y)} \tag{11}$$

or $X = 2\sqrt{y(h - y)}$ in dimensional units. This result agrees with the range of a ball rolling off a horizontal tabletop to the floor, as obtained by eliminating t between the two equations $h - y = 0.5gt^2$ and $X = (2gy)^{1/2}t$. More generally, if $|y'| \ll 1$ everywhere along the curve, as describes a gentle pitch, then equation (11) should hold to a good approximation. In that case, maximum range X is obtained if the release point is midway down the curve at $y = 1/2$.

3.2. Particle launched from a linear ramp

The total range R is the horizontal distance from the origin,

$$R = x + 2\frac{\sqrt{y(1 - y + y'^2)} - yy'}{1 + y'^2}. \tag{12}$$

Maximize it by setting its derivative to zero,

$$\frac{\mathrm{d}R}{\mathrm{d}x} = 1 + 2y'\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\sqrt{y(1 - y + y'^2)} - yy'}{1 + y'^2}\right] = 0. \tag{13}$$

For a linear ramp $y = kx$ with $k > 0$, equation (13) becomes

$$k(1 + k^2 - 2y_{\max}) = (k^2 - 1)\sqrt{y_{\max}(1 + k^2 - y_{\max})}. \tag{14}$$

Squaring both sides and rearranging, one finds

$$y_{\max}^2 - (1 + k^2)y_{\max} + k^2 = 0. \tag{15}$$

This quadratic equation has two roots, $y_{\max} = 1$ and $y_{\max} = k^2$. However, if these two roots of equation (15) are substituted back into equation (14), only $y_{\max} = 1$ is found to be a solution of it. The other root $y_{\max} = k^2$ results in the two sides of equation (14) differing in their signs. (That sign difference was eliminated during the subsequent squaring.) The conclusion is that the total range R is maximized by not releasing the particle at all and instead waiting until the roller coaster reaches ground level at $y = 1$.
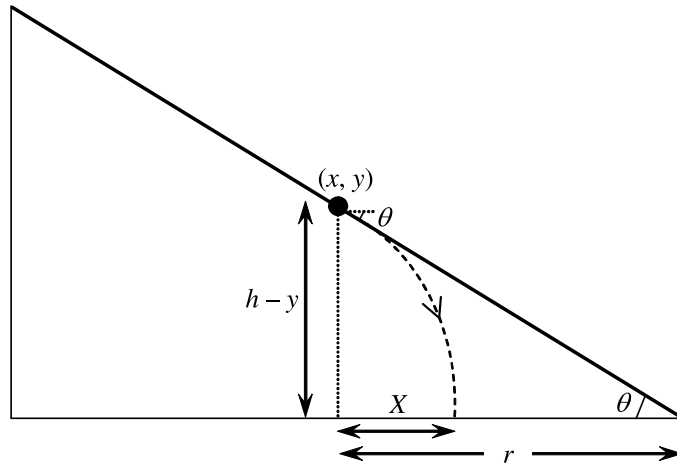
Figure 2. The particle travels horizontally a distance X after being dropped out of the cart at point (x, y). On the other hand, if the particle had stayed in the cart until the ramp reached ground level, the particle would have travelled a horizontal distance r.

Here is an alternative direct proof of this conclusion. If the particle is released at point (x, y) with speed υ, it will travel an additional horizontal distance X before it hits the ground, as sketched in figure 2. On the other hand, if it stays on the ramp, it will instead travel an additional horizontal distance of $r = (h - y) \cot \theta$ before reaching the ground. By eliminating t between equations (1) and (2), one finds

$$h - y = X \tan \theta + \frac{g}{2 \upsilon^2 \cos^2 \theta} X^2. \tag{16}$$

Express the left-hand side in terms of r and divide through by $\tan \theta$ to obtain

$$r = X + BX^2 \quad \text{where} \quad B \equiv \frac{g}{\upsilon^2 \sin 2\theta} > 0. \tag{17}$$

Since BX^2 is positive, one immediately sees that $r > X$. Physically, if one releases the particle its subsequent trajectory will fall increasingly below the ramp (as the dashed curve in figure 2 indicates) and hence its horizontal range will be reduced.

So consider maximizing X rather than R, by setting the y derivative of the numerator of the quantity in the square brackets in equation (13) to zero. Then one finds

$$1 + k^2 - 2y_{\max} = 2k \sqrt{y_{\max}(1 + k^2 - y_{\max})}. \tag{18}$$

To solve this equation, again square it, find the two roots of the resulting quadratic equation, and reject the root that is not a solution of equation (18). The final result is

$$y_{\max} = \frac{1 + k^2 - k\sqrt{1 + k^2}}{2}. \tag{19}$$

When k is near zero in value, then $y_{\max} \approx 1/2$, the same result as found following equation (11). When k is large, then

$$y_{\max} = \frac{1 + k^2 - k^2\sqrt{1 + k^{-2}}}{2} \approx \frac{1 + k^2 - k^2 \left(1 + \frac{1}{2}k^{-2}\right)}{2} = \frac{1}{4}. \tag{20}$$

As the slope of the ramp increases, the release point for maximum range X monotonically decreases from halfway to a quarter of the way vertically down the ramp. For a $45°$ ramp, when $k = 1$, the particle should be released at $y_{\max} = 1 - 2^{-1/2} \approx 0.293$.

3.3. Particle launched from a parabola

For a parabolic curve that starts at the origin with zero slope, $y = Ax^2$ with $A > 0$, the equation determining the maximum value of X has one positive real solution for x. That equation simplifies to a quadratic when the term in parentheses in equation (7) equals unity, i.e. when $y = y'^2$. Substituting $y = Ax^2$ into this differential equation, one finds $A = 1/4$. For that value of A, equation (7) becomes

$$X = 2\sqrt{y}\frac{1-y}{1+y} \tag{21}$$

which is a maximum when its derivative is zero. That occurs when

$$y_{max}^2 + 4y_{max} - 1 = 0 \tag{22}$$

whose positive solution is

$$y_{max} = \sqrt{5} - 2 \tag{23}$$

3.4. Particle launched from a catenary

Consider the hyperbolic arch that starts at the origin with zero slope and unit curvature [2], $y = \cosh x - 1$. Equation (7) becomes

$$X = 2\frac{\sinh x(1 - \cosh x) + \sqrt{\cosh^3 x - 2\cosh^2 x + 2\cosh x - 1}}{\cosh^2 x}. \tag{24}$$

Take the x derivative of this equation and set it to zero. After some algebra, one obtains

$$4(\cosh x + 2)^2(\cosh x - 1)^2(\cosh^2 x - \cosh x + 1)$$
$$= (\cosh x + 1)(\cosh x - 2)^2(\cosh^2 x - 2\cosh x + 2)^2 \tag{25}$$

which rearranges into a quartic equation

$$\cosh^4 x - 11\cosh^3 x + 16\cosh^2 x - 12\cosh x = -8 \tag{26}$$

with two real solutions. Only one of these two solutions corresponds to a positive flight time, namely release at coordinates $(x, y) \approx (0.5894, 0.1788)$.

4. Computer solution for other shapes of tracks

Instructors are invited to have students explore other curves. Many shapes can be inverted or the origin can be shifted, so that any given function affords numerous possibilities. Here are a few suggested tracks to try.

- Roll a marble down the interior of a semicircular track resting on the ground, so that $y(x) = (2x - x^2)^{1/2}$, where a point diametrically across from the starting point would be reached at $x_f = 2$. Alternatively, this geometry could be that of a person swinging on the end of a rope like a simple pendulum and then releasing it and flying through the air [3].
- Reminiscent of a standard textbook problem [4], consider a pebble starting at the top of a hemispherical snowball such that $y(x) = 1 - (1 - x^2)^{1/2}$ and sliding down it until it flies off the surface. The snowball makes contact with the ground at $x_f = 1$.
- A rope hangs in the shape of a catenary such as $y(x) = \cosh(1) - \cosh(x - 1)$, suspended between a post at the origin and another post at $x_f = 2$. One could consider a zipline rider [5] travelling down this rope and letting go above the surface of a pond.
- Cycloids and lemniscates arise in the problem of calculating the time of descent of a bead sliding frictionlessly down a wire [6].

- Roller coaster loop-the-loops have a characteristic teardrop shape described by functions such as a clothoid [7] to minimize any abrupt changes in the centripetal acceleration of the riders.

The analysis can be performed numerically using computer mathematical software. For example, here is the complete Mathematica code to analyse the first bulleted example above, including two plots that help visualize the key functions:

$$y[x\_]:=\text{Sqrt}[2x-x^2]$$
$$\text{Plot}[y[x], \{x, 0, 2\}]$$
$$X[x\_]:=2(\text{Sqrt}[y[x](1-y[x]+y'[x]^2)]-y[x]y'[x])/(1+y'[x]^2)$$
$$\text{Plot}[X[x], \{x, 0, 2\}]$$
$$\text{NSolve}[X'[x]==0,x]$$

In the last line, it is often faster to use the command FindRoot instead of NSolve, but in that case one must replace the final x in that expression with $\{x,n\}$ where n is a value of x near the maximum value of X[x] of interest, as can be found by inspection of the second plot produced by this code. The results can be verified by preparing a table of (x, y) values for positions along a specified curve in a spreadsheet such as Excel in appropriately fine steps such as $\Delta x = 0.01$. For each point in this table, one can calculate the angle of incline of the track either approximately as $\theta = \tan^{-1}(\Delta y/\Delta x)$ or exactly as $\theta = \tan^{-1}(\mathrm{d}y/\mathrm{d}x)$. By substituting both θ and y into equation (5), one can then compute and plot X and compare it to the second graph generated by Mathematica above. The approximate coordinates $(x_{\max}, y_{\max})$ for maximum X can be obtained by examining these computed values of X.

5. Conclusions

This article has analysed the maximum range of a particle that falls off a frictionless track of known shape. Analytic solutions have been found and discussed for launch off a horizontal ramp, an inclined plane, a parabola, and a hyperbolic arch. Additional curves can easily be explored numerically with the assistance of a computer, either using a computational program such as Mathematica, or at a simpler level by using a spreadsheet such as Excel. An interesting student project might then consist in using a commercial lab-scale roller-coaster system (such as the one sold by PASCO) and magnetically releasing a small ball bearing from it to see how far it flies through the air before landing on the floor.

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